

3+1 Approach to the Long Wavelength Iteration Scheme

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ABSTRACT

Large-scale inhomogeneities and anisotropies are modeled using the Long Wavelength Iteration Scheme. In this scheme solutions are obtained as expansions in spatial gradients, which are taken to be small. It is shown that the choice of foliation for spacetime can make the iteration scheme more effective in two respects: (i) the shift vector can be chosen so as to dilute the effect of anisotropy on the late-time value of the extrinsic curvature of the spacelike hypersurfaces of the foliation; and (ii) pure gauge solutions present in a similar calculation using the synchronous gauge vanish when the spacelike hypersurfaces have extrinsic curvature with constant trace. We furthermore verify the main conclusion of the synchronous gauge calculation which is large-scale inhomogeneity decays if the matter—considered to be that of a perfect-fluid with a barotropic equation of state—violates the strong-energy condition. Finally, we obtain the solution for the lapse function and discuss its late-time behaviour. It is found that the lapse function is well-behaved when the matter violates the strong energy condition.

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1. Introduction

The long wavelength iteration scheme [1,2,3] has been used to model analytically cosmologies with large scale inhomogeneity. This scheme was first introduced by Lifschitz and Khalatnikov (see [4] and references therein). A similar scheme has been used by Salopek, Stewart, and collaborators [5] in their analysis of the Hamilton-Jacobi equation for general relativity. It has also been applied to Brans-Dicke theory [6] and the more general scalar-tensor gravity [7]. The essential part of the long wavelength iteration scheme is spatial gradients are considered to be small. Therefore, the scheme immediately suffers from the fact that there is no absolute space, and hence no absolute notion of spatial gradient, large or small. We circumvent this difficulty by using the 3+1 formalism to perform our calculations (see York [8] for an excellent review). The technical problems associated with the lack of an absolute space are overcome by introducing a foliation of spacetime, and hence a notion of time and the flow of time. But furthermore, we will see that the 3+1 formalism can also make the long wavelength iteration scheme more effective at producing inhomogeneous and anisotropic cosmological models.

Refs. [1,2,3] solve a first-order (*i.e.*, one time derivative) form of the field equations where the synchronous gauge is imposed. In the language of the 3+1 formalism, the synchronous gauge sets the shift vector to zero and the lapse function to one. But the immediate simplification this gauge choice obtains, in making the field equations less complicated by having fewer fields to worry about, does not necessarily make the calculations less complicated (see, for instance, Piran [9], Smarr and York [10], or Hawking and Ellis [11]). For instance, it will be shown below that the shift vector can be used to exponentially damp (in time) the influence of anisotropy on the value of the extrinsic curvature of the spacelike leaves of the foliation. This can be significant because in the synchronous gauge approach obtaining the three-metric of the spacelike slices of the foliation when anisotropy is present is complicated and was only treated as a linear perturbation during the iterative part of the calculations.

The most important difference between the formulation used here and that of Ref. [1] is in the choice of lapse function: we will use a lapse function such that the trace of

the extrinsic curvature, K , of each spacelike slice associated with the foliation is constant. The value of K will change from one slice to the next, and is thus a function of time, *i.e.*, $K = k(t)$. (The trace can even be taken to be the time itself [12], which is known as ‘York time.’) The $K = k(t)$ slicing condition is used because it ensures better than the synchronous gauge that calculated inhomogeneity is physical [12,13,14], and not merely an artifact of embedding unnecessarily distorted spacelike slices into an otherwise homogeneous spacetime. Indeed, it will be seen below that pure gauge solutions appearing in the results of Comer *et al.* [1] do not show up here. However, we do find their other solutions and thus confirm the physical origin of the inhomogeneity they calculate. Furthermore, we also confirm their main conclusion, which is long wavelength inhomogeneity grows or decays, depending on the choice of the equation of state for the matter. In particular, the inhomogeneity decays when the matter is ‘inflationary’ (*i.e.*, it violates the strong energy condition [11]).

It must be stated now that we do not solve the 3+1 form of the field equations as an initial-value problem. That is, no explicit attempt is made to solve first the constraints—so as to isolate the true degrees of freedom, whose initial values make up the initial data set—and then use the remaining field equations to evolve the initial data set forward in time. Here expansions in terms of spatial gradients of a “seed” metric are made for the three-metric of the spacelike slices and matter variables. The zeroth-order term in an expansion contains no spatial gradients, the second-order term, which is next, contains two spatial derivatives, the fourth-order has four, and so on. (We shall demonstrate that the lapse function, but not the shift vector, must also be expanded in this way.) The coefficients in the expansions are constructed so that the constraints, as well as the other equations, are satisfied order-by-order. In this context, the initial-value problem becomes just a different way of solving the field equations; however, we will discuss it briefly in the concluding section.

Building a slicing of spacetime based on keeping the trace of the extrinsic curvature constant has now a long history, beginning with the maximal slicing condition of Lichnerowicz [15]. Since then much work has been done to determine when the $K = k(t)$ condition may be used, for instance, in the context of homogeneous cosmology (see Ryan

and Shepley [16] for a review) or vacuum, asymptotically flat spacetimes with no restriction to homogeneity [17]. Goddard [18] has even shown for closed universes that if one $K = k(t)$ slice exists, and the strong energy condition is satisfied, then the $K = k(t)$ condition may be applied through the whole spacetime. More recently, Goldwirth and Piran [19] demonstrated numerically that the $K = k(t)$ slicing condition breaks down for closed, inhomogeneous universes having ‘inflationary’ matter. What they find is one cannot prescribe a $k(t)$ which is a monotonic function of time and simultaneously construct an acceptable lapse function. However, they were able to use $K = k(t)$ for open universes, even during inflation, with no problems. As for the present work, there are no difficulties with using the $K = k(t)$ slicing condition when the matter is ‘inflationary.’ We can specify a monotonic $k(t)$ and still get an acceptable solution for the lapse function (in the sense that it remains positive for all time). On the other hand, when the matter satisfies the strong energy condition, caution must be exercised in the use of $K = k(t)$, especially when the higher-order terms in the lapse function become comparable to the zeroth-order term.

In Sec. 2 we give the 3+1 form of the field equations and set the bulk of the notation. In Sec. 3 we discuss in general terms (*i.e.*, for arbitrary lapse function and shift vector) the long wavelength approximation [20]. In Sec. 4, we discuss the zeroth-, second-, and fourth-order equations and solutions for $K = k(t)$ hypersurfaces. In Sec. 5, we write the solution for the lapse function, good to fourth-order in spatial gradients, and discuss how it behaves at late times. In Sec. 6, we offer some concluding remarks. Finally, some formulas used to construct the second- and fourth-order equations are listed in Appendix A and the method through which the zeroth-order solution for the extrinsic curvature is obtained is given in Appendix B. We will use units throughout such that $G = c = 1$ as well as ‘MTW’ conventions (see Ref. [21]).

2. The 3+1 Decomposition and Field Equations

The notion of ‘time’ in the 3+1 formalism enters via a scalar function $t = t(x^\mu)$, $\mu = 0, 1, 2, 3$, whose level surfaces are spacelike. The flow of time is introduced via a vector t^μ , which can be either timelike or spacelike. The vector which is perpendicular to the level surfaces of $t(x^\mu)$ is $n_\mu = -N\nabla_\mu t$, where the lapse function N ensures that $n^\nu n_\nu = -1$.

The vector t^μ is not parallel to n^μ but has the form

$$t^\nu = (\perp_\mu^\nu - n^\nu n_\mu) t^\mu \equiv N^\nu + N n^\nu , \quad (1)$$

where $\perp_\mu^\nu = \delta_\mu^\nu + n^\nu n_\mu$ is the ‘projection-operator’ that projects spacetime tensors into the level surfaces obtained from $t(x^\mu)$ and N^μ is the shift vector ($n_\nu N^\nu = 0$). From Eq. (1), it is seen that N must always remain positive, otherwise there would exist spacetime points where t^μ would be parallel to the spacelike leaves of the foliation [10]. It must also be emphasized that the lapse function and shift vector are arbitrary in the sense that the four independent functions associated with them are not determined by the field equations but instead must be specified by hand.

From here on out it will be assumed that a local coordinate system $x^\mu = (x^0, x^i)$ exists, where the x^i , $i = 1, 2, 3$, are the coordinates of the spacelike slices associated with the foliation and we take $t = x^0$. With respect to these coordinates $N^\mu = (0, N^i)$, and the line interval can be written

$$ds^2 = -(N^2 - N^i N_i) dt^2 + 2N_i dx^i dt + \gamma_{ij} dx^i dx^j , \quad (2)$$

where $\gamma_{ij} = g_{ij}$ and $N_i = \gamma_{ij} N^j$. The spatial covariant derivative compatible with γ_{ij} is denoted \tilde{D}_i .

The 3+1 field equations consist of four constraint equations and twelve evolution equations (see Ref. [8] for more discussion). They contain terms with only one time derivative because $\dot{\gamma}_{ij}$ (where a dot overscript ‘ $\dot{}$ ’ means $\partial/\partial t$) is replaced by the extrinsic curvature, whose spatial components K_{ij} are obtained from [22]

$$K_{ij} = -\frac{1}{2} \perp_i^\sigma \perp_j^\tau \nabla_\sigma n_\tau = \frac{1}{2} N^{-1} \left[\tilde{D}_i N_j + \tilde{D}_j N_i - \dot{\gamma}_{ij} \right] . \quad (3)$$

The inverted form of Eq. (3) thus represents six of the evolution equations (for γ_{ij}). The other evolution equations, and the constraints, result from projecting the free indices of the Einstein equations into and out of the spacelike slices of the foliation. The three independent projections of the stress-energy tensor $T_{\mu\nu}$ define an energy density $\rho \equiv n^\mu n^\nu T_{\mu\nu}$, a three-current $\mathcal{J}_i \equiv -n^\nu \perp_i^\sigma T_{\nu\sigma}$, and spatial stress tensor $\mathcal{S}_{ij} \equiv \perp_i^\sigma \perp_j^\tau T_{\sigma\tau}$. The 3+1 equations are

$$16\pi\rho = \tilde{R} - K_j^i K_i^j + K^2 , \quad (4)$$

$$8\pi\mathcal{J}_i = \tilde{D}_j \left(K_i^j - K\delta_i^j \right) , \quad (5)$$

$$8\pi\mathcal{S}_i^j = \tilde{R}_i^j + N^{-1} \left(K_k^j \tilde{D}_i N^k - K_i^k \tilde{D}_k N^j \right) + K K_i^j - N^{-1} \dot{K}_i^j - \\ N^{-1} \tilde{D}_i \tilde{D}^j N + N^{-1} N^k \tilde{D}_k K_i^j - 4\pi(\rho - \mathcal{S})\delta_i^j , \quad (6)$$

and

$$\dot{\gamma}_{ij} = \tilde{D}_i N_j + \tilde{D}_j N_i - 2N K_{ij} , \quad (7)$$

where \tilde{R}_{ij} is the three-dimensional Ricci tensor formed from γ_{ij} , $\tilde{R} = \gamma^{ij} \tilde{R}_{ij}$ is its associated Ricci scalar, and $\mathcal{S} \equiv \gamma^{ij} \mathcal{S}_{ij}$. Note that Eq. (6) is the trace free form of $\perp_i^\sigma \perp_j^\tau G_{\sigma\tau} = 8\pi\mathcal{S}_{ij}$, the removed trace being

$$\dot{K} = 4\pi N (\rho + \mathcal{S}) + N^i \tilde{D}_i K + N K_i^j K_j^i - \tilde{D}^i \tilde{D}_i N . \quad (8)$$

Lastly, the evolution equation for the determinant of the three-metric γ (which will be used below) is

$$\dot{\gamma} = 2\gamma \left(\tilde{D}_i N^i - N K \right) . \quad (9)$$

The field equations for a perfect fluid are $\nabla_\nu T_\mu^\nu = 0$. The independent projections of the free index gives

$$\dot{\rho} + N \tilde{D}_i \mathcal{J}^i = N \left(K_j^i \mathcal{S}_i^j + K \rho \right) - 2\mathcal{J}^i \tilde{D}_i N + N^i \tilde{D}_i \rho \quad (10)$$

and

$$\dot{\mathcal{J}}^i + N \tilde{D}_j \mathcal{S}^{ij} = N \left(2K_j^i \mathcal{J}^j + K \mathcal{J}^i \right) - \mathcal{S}^{ij} \tilde{D}_j N - \rho \tilde{D}^i N + N^j \tilde{D}_j \mathcal{J}^i - \mathcal{J}^j \tilde{D}_j N^i , \quad (11)$$

where ρ , \mathcal{J}^i and \mathcal{S}_{ij} are obtained from the projections of the perfect fluid stress-energy tensor

$$T_{\mu\nu} = (\rho^* + p^*) u_\mu u_\nu + p^* g_{\mu\nu} . \quad (12)$$

The fluid four-velocity u^μ has the normalization $u^\mu u_\mu = -1$ and the pressure p^* is related to the energy density ρ^* via the equation of state $p^* = (\Gamma - 1)\rho^*$, where $0 \leq \Gamma \leq 2$. Letting \tilde{u}^i represent the projection of the four-velocity into the slices of the foliation, *i.e.*, $\tilde{u}^i = \perp_\sigma^i u^\sigma$, and $|\tilde{\mathbf{u}}|^2 \equiv \tilde{u}^i \tilde{u}_i$, then

$$\rho = \rho^* (1 + \Gamma |\tilde{\mathbf{u}}|^2) , \quad (13)$$

$$\mathcal{J}_i = \pm \Gamma \rho^* \sqrt{1 + |\tilde{\mathbf{u}}|^2} \tilde{u}_i , \quad (14)$$

and

$$\mathcal{S}_{ij} = \Gamma \rho^* \tilde{u}_i \tilde{u}_j + (\Gamma - 1) \rho^* \gamma_{ij} . \quad (15)$$

3. The Long Wavelength Approximation

In this section we invoke the long wavelength approximation, where spatial gradients are neglected entirely. These solutions are accurate representations of cosmologies wherein deviation from perfect homogeneity occurs on scales bigger than the co-moving Hubble length (see Tomita [20], and also Ref. [1], for more discussion). It will be seen that one can extract a significant amount of information about the solutions before any assumptions about the lapse function are made and without setting the shift vector to zero. In particular, since the lapse function and shift vector are specified freely, there is no a priori reason for their spatial gradients to vanish.

The long wavelength equations are

$$16\pi\rho_0 \approx -^{(0)}K_j^i {}^{(0)}K_i^j + {}^{(0)}K^2 , \quad (16)$$

$$8\pi {}^{(0)}\mathcal{J}_i \approx 0 , \quad (17)$$

and

$$\begin{aligned} 8\pi {}^{(0)}\mathcal{S}_i^j \approx & N^{-1} \left({}^{(0)}K_k^j \tilde{D}_i N^k - {}^{(0)}K_i^k \tilde{D}_k N^j \right) + {}^{(0)}K {}^{(0)}K_i^j - N^{-1} {}^{(0)}\dot{K}_i^j - \\ & N^{-1} \tilde{D}_i \tilde{D}^j N - 4\pi(\rho_0 - {}^{(0)}\mathcal{S})\delta_i^j , \end{aligned} \quad (18)$$

where a ‘0’ left-superscript (or right-subscript) on any quantity signifies it is zeroth-order in spatial gradients. The equations for $\dot{\gamma}_{ij}$ and $\dot{\gamma}$ do not change their form from the relations given in the last section; however, that for \dot{K} does change to

$${}^{(0)}\dot{K} \approx 4\pi N \left(\rho_0 + {}^{(0)}\mathcal{S} \right) + N {}^{(0)}K_i^j {}^{(0)}K_j^i - \tilde{D}^i \tilde{D}_i N . \quad (19)$$

The stress-energy divergence equations become, by forcing Eq. (17) to hold on each space-like slice (which means taking ${}^{(0)}\tilde{u}_i \approx 0$ at all times),

$$\dot{\rho}_0 \approx N \left({}^{(0)}K_j^i {}^{(0)}\mathcal{S}_i^j + {}^{(0)}K \rho_0 \right) \quad (20)$$

and

$$0 \approx \left({}^{(0)}\mathcal{S}_i^j + \rho_0 \delta_i^j \right) \tilde{D}_j N . \quad (21)$$

Two immediate conclusions follow from Eqs. (20) and (21): using the evolution equation for γ , then it can be obtained from Eq. (20) that

$$\rho_0 {}^{(0)}\gamma^{\Gamma/2} \approx \mathcal{C} \exp \left(\Gamma \int_{t_0}^t d\tilde{t} \tilde{D}_i N^i \right) , \quad (22)$$

where \mathcal{C} depends only on x^i . Eq. (21) says that $\tilde{D}_i N \approx 0$ whenever the combination ${}^{(0)}\mathcal{S}_i^j + \rho_0 \delta_i^j$ is not zero. (For a perfect fluid with the equation of state used here, the combination is zero only when $\Gamma = 0$.) Hence, even though the lapse function is in principle freely specifiable, consistency of the long wavelength approximation demands that it have a small spatial gradient.

The remaining equations (18) and (7) are to be solved, respectively, to obtain ${}^{(0)}K_i^j$ and then ${}^{(0)}\gamma_{ij}$. Finding ${}^{(0)}K_i^j$ requires a rewriting of Eq. (18). This is accomplished by using Eq. (16), and the fact that $\tilde{D}_i N$ and the three-velocity are both zero at this order. Also letting ${}^{(0)}\mathcal{K}_i^j \equiv \sqrt{{}^{(0)}\gamma} \left[{}^{(0)}K_i^j - \frac{1}{3} {}^{(0)}K \delta_i^j \right]$, which is the trace-free part of ${}^{(0)}K_i^j$ (modulo the $\sqrt{{}^{(0)}\gamma}$ factor), then Eq. (18) becomes

$${}^{(0)}\dot{\mathcal{K}}_i^j - \mathcal{L}_{\mathbf{N}} {}^{(0)}\mathcal{K}_i^j \approx 0 , \quad (23)$$

where $\mathcal{L}_{\mathbf{N}} {}^{(0)}\mathcal{K}_i^j$ is the Lie derivative with respect to N^i , *i.e.*, ignoring spatial gradients of the extrinsic curvature and the three-metric,

$$\mathcal{L}_{\mathbf{N}} {}^{(0)}\mathcal{K}_i^j \approx \left(\tilde{D}_k N^k \right) {}^{(0)}\mathcal{K}_i^j + \left(\delta_l^j \tilde{D}_i N^k - \delta_i^k \tilde{D}_l N^j \right) {}^{(0)}\mathcal{K}_k^l . \quad (24)$$

In order to solve Eq. (23), it is convenient to interpret it as a linear vector equation (with the vector \mathbf{K} being formed out of the components of ${}^{(0)}\mathcal{K}_i^j$), *i.e.*,

$$\dot{\mathbf{K}} - \left(\tilde{D}_k N^k \right) \mathbf{K} - \mathbf{M} \mathbf{K} \approx 0 , \quad (25)$$

where

$$\mathbf{K} = [{}^{(0)}\mathcal{K}_1^1 \ {}^{(0)}\mathcal{K}_2^1 \ {}^{(0)}\mathcal{K}_3^1 \ {}^{(0)}\mathcal{K}_1^2 \ {}^{(0)}\mathcal{K}_2^2 \ {}^{(0)}\mathcal{K}_3^2 \ {}^{(0)}\mathcal{K}_2^3 \ {}^{(0)}\mathcal{K}_2^3 \ {}^{(0)}\mathcal{K}_3^3]^T \quad (26)$$

and

$$\mathbf{M} = \begin{bmatrix} 0 & a_4 & a_7 & -a_2 & 0 & 0 & -a_3 & 0 & 0 \\ a_2 & a_5 - a_1 & a_8 & 0 & -a_2 & 0 & 0 & -a_3 & 0 \\ a_3 & a_6 & a_9 - a_1 & 0 & 0 & -a_2 & 0 & 0 & -a_3 \\ -a_4 & 0 & 0 & a_1 - a_5 & a_4 & a_7 & -a_6 & 0 & 0 \\ 0 & -a_4 & 0 & a_2 & 0 & a_8 & 0 & -a_6 & 0 \\ 0 & 0 & -a_4 & a_3 & a_6 & a_9 - a_5 & 0 & 0 & -a_6 \\ -a_7 & 0 & 0 & -a_8 & 0 & 0 & a_1 - a_9 & a_4 & a_7 \\ 0 & -a_7 & 0 & 0 & -a_8 & 0 & a_2 & a_5 - a_9 & a_8 \\ 0 & 0 & -a_7 & 0 & 0 & -a_8 & a_3 & a_6 & 0 \end{bmatrix}, \quad (27)$$

with $a_1 = \tilde{D}_1 N^1$, $a_2 = \tilde{D}_2 N^1$, $a_3 = \tilde{D}_3 N^1$, $a_4 = \tilde{D}_1 N^2$, $a_5 = \tilde{D}_2 N^2$, $a_6 = \tilde{D}_3 N^2$, $a_7 = \tilde{D}_1 N^3$, $a_8 = \tilde{D}_2 N^3$, and $a_9 = \tilde{D}_3 N^3$. One can quickly verify that $\text{Tr} \mathbf{M} = 0$. Therefore, the eigenvalues of \mathbf{M} must add up to zero.

Even though Eq. (25) is linear in the extrinsic curvature, solving it for a general shift vector is a daunting linear algebra eigenvalue problem, further complicated by the time dependence in the matrix \mathbf{M} . We will simplify the problem by using the freedom to specify, by hand, the shift vector and assume that its spatial gradient is time-independent. Nevertheless, finding the eigenvalues of the matrix \mathbf{M} still means solving a ninth-order polynomial equation. Fortunately, a lot of progress can be made even without knowing explicitly the eigenvalues. By following the procedure outlined in Appendix B, then it can be shown that

$${}^{(0)}\mathcal{K}_i^j = \exp \left[\left(\tilde{D}_k N^k \right) t \right] \mathcal{G}_i^j. \quad (28)$$

The tensor $\mathcal{G}_i^j(x^k, t)$ is the analog of the so-called anisotropy matrix obtained in the synchronous gauge calculations of Ref. [1]. The big difference between our result and that of Ref. [1] is the anisotropy matrix of Ref. [1] is independent of time whereas \mathcal{G}_i^j has an exponential time dependence. (Clearly, \mathcal{G}_i^j becomes time-independent when the shift vector is zero.) Hence, we see explicitly that one can artificially remove or enhance the anisotropy coming from \mathcal{G}_i^j , depending on what choice is made for the shift vector. More to the point, let us consider the simplified example where only a_1 , a_5 , and a_9 are non-zero, with $a_1 = a_5 = a_9 < 0$. Then it is easy to see that ${}^{(0)}\mathcal{K}_i^j$ decays to zero exponentially with time. Hence, as t increases, it will quickly be the case that

$${}^{(0)}K_i^j = \frac{1}{3} {}^{(0)}K \delta_i^j. \quad (29)$$

The last item for this section is to determine $^{(0)}\gamma_{ij}$. Since we have seen how to use the shift vector to reduce the effect of $\mathcal{G}_i^j(x^k, t)$ on $^{(0)}\mathcal{K}_i^j$, we will henceforth set both N^i and \mathcal{G}_i^j to zero for the rest of this work. Eq. (29) thus becomes the zeroth-order solution for the extrinsic curvature. From Eqs. (7), (9), and (29) it follows that

$$^{(0)}\gamma_{ij} = ^{(0)}\gamma^{1/3} h_{ij} \quad , \quad h^{jk} h_{ki} = \delta_i^j \quad , \quad (30)$$

where h_{ij} depends only on x^i and is the so-called “seed” metric mentioned in the introduction. The remaining field equations only give one more thing, when Eq. (22) is used, and that is

$$^{(0)}K = -\sqrt{24\pi\mathcal{C}} \ ^{(0)}\gamma^{-\Gamma/4} \quad . \quad (31)$$

To determine the explicit time-dependence of $^{(0)}\gamma$, and thus the other zeroth-order quantities, the lapse function must be specified. For instance if $N = 1$, like in the synchronous gauge, then $^{(0)}\gamma^{1/3} \propto t^{4\Gamma/3}$. Or, a condition like $K = k(t)$ must be imposed, which is what we do next.

4. The Long Wavelength Iteration Scheme

The long wavelength iteration scheme obtains solutions to the Einstein equations by assuming spatial gradients of the gravitational and matter fields are small on every spacelike slice of the foliation. The solutions contain terms with ever-increasing spatial gradients of the “seed” metric obtained in the last section. To have solutions with inhomogeneity on shorter and shorter scales, terms with more and more spatial gradients must be included [23]. Here we write down solutions with terms containing up to four gradients.

To be precise, the field variables ρ , \tilde{u}^i , and γ_{ij} are expanded as follows (letting $A_0(t) \equiv ^{(0)}\gamma^{1/3}$):

$$\rho = \rho_0(t) + \rho_2(t)R + \rho_4 R^2 + \mu_4(t)R_{ij}R^{ij} + \epsilon_4 D_i D^i R + \dots \quad , \quad (32)$$

$$\begin{aligned} \tilde{u}_i = & u_3(t)D_i R + u_5(t)R D_i R + v_5(t)R_i^j D_j R + w_5(t)R_k^j D_i R_j^k + \\ & x_5(t)R_j^k D_k R_i^j + y_5(t)D_i D_j D^j R \dots \quad , \end{aligned} \quad (33)$$

and

$$\gamma_{ij} = ^{(0)}\gamma_{ij} + ^{(2)}\gamma_{ij} + ^{(4)}\gamma_{ij} + \dots \quad , \quad (34)$$

where R_{ij} is the Ricci tensor obtained using the “seed” metric h_{ij} , $R = h^{ij}R_{ij}$, D_i is the covariant derivative compatible with the “seed” h_{ij} ,

$$^{(0)}\gamma_{ij} = A_0(t)h_{ij} , \quad (35)$$

$$^{(2)}\gamma_{ij} = A_0(t) \left[f_2(t)R_{ij} + \frac{1}{3} (g_2(t) - f_2(t)) Rh_{ij} \right] , \quad (36)$$

and

$$\begin{aligned} ^{(4)}\gamma_{ij} = A_0(t) & \left[\frac{1}{3} (a_4(t) - b_4(t)) R^2 h_{ij} + b_4(t) R R_{ij} + \frac{1}{3} (c_4(t) - d_4(t)) R^{kl} R_{kl} h_{ij} + \right. \\ & d_4(t) R_j^k R_{ki} + \frac{1}{3} (e_4(t) - f_4(t) - g_4(t)) (D_k D^k R) h_{ij} + f_4(t) D_i D_j R + \\ & \left. g_4(t) D_k D^k R_{ij} \right] . \end{aligned} \quad (37)$$

The higher order terms in the expansions above are “small” in the sense that R_{ij} contains two spatial gradients. That is, if L is the characteristic scale on which the fields vary, then $R_{ij} \sim L^{-2} h_{ij}$, $D_i D_j R_{kl} \sim L^{-4} h_{ij} h_{kl}$, and so on.

When the $K = k(t)$ slicing condition is imposed, the lapse function is no longer freely specified, but is obtained from

$$N = -\frac{1}{k(t)} \frac{\dot{\sqrt{\gamma}}}{\sqrt{\gamma}} , \quad (38)$$

once $k(t)$ is given and the form of γ is known. The 3+1 field equations also reflect the difference, becoming

$$16\pi\rho = \tilde{R} - K_j^i K_i^j + k^2(t) , \quad (39)$$

$$8\pi\mathcal{J}_i = \tilde{D}_j K_i^j , \quad (40)$$

$$8\pi\mathcal{S}_i^j = \tilde{R}_i^j + k(t) K_i^j + \frac{k(t)}{(\ln\sqrt{\gamma})} \dot{K}_i^j - \frac{1}{(\ln\sqrt{\gamma})} \tilde{D}_i \tilde{D}^j (\ln\sqrt{\gamma}) - 4\pi (\rho - \mathcal{S}) \delta_i^j , \quad (41)$$

and

$$\dot{\gamma}_{ij} = \frac{2}{k(t)} \frac{\dot{\sqrt{\gamma}}}{\sqrt{\gamma}} K_{ij} . \quad (42)$$

Determining N from Eq. (38) is unconventional. However, it will be shown to be equivalent to the conventional approach (which is to solve Eq. (8)) in the next section.

The ordinary differential equations that determine the time-dependent coefficients in Eqs. (32-34) come about by putting the expansions in Eqs. (32-34) into the 3+1 equations

just above, and then rearranging until all “like” terms (*i.e.*, quantities containing h_{ij} , Rh_{ij} , or R_{ij} , *etc.*) are gathered. One then gets as many equations as coefficients, since at each order the tensors Rh_{ij} , R_{ij} , *etc.*, are in general linearly independent. We will not list the zeroth-order equations and solutions here since they are the same as those presented in the last section (except that $^{(0)}K$ is to be replaced with $k(t)$). The second-order equations for $f_2(t)$ and $g_2(t)$ are

$$\left[k^{2(\Gamma-1)/\Gamma} f_2' \right]' = -\frac{8}{\Gamma^2 A_0(k)} k^{-2(1+\Gamma)/\Gamma} \quad (43)$$

and

$$kg_2' = -\frac{2(2-3\Gamma)}{\Gamma^2 k^2 A_0(k)}, \quad (44)$$

where $' = d/dk = \dot{k}^{-1} d/dt$ and $A_0(k) \propto k^{-4/3\Gamma}$. The fourth-order equations are

$$ka_4' = \frac{2(3\Gamma-2)}{3\Gamma^2 k^2 A_0(k)} (f_2 - g_2) - \frac{2-\Gamma}{8A_0(k)} (kf_2')^2 - \frac{k}{3} (f_2 f_2' - g_2 g_2') - \frac{(2-3\Gamma)^2}{\Gamma^3 k^4 A_0^2(k)}, \quad (45)$$

$$\begin{aligned} \left[k^{2(\Gamma-1)/\Gamma} \Sigma_4' \right]' &= \frac{2-3\Gamma}{\Gamma^2 A_0(k)} k^{-2(1+\Gamma)/\Gamma} (kf_2') + \frac{4}{3\Gamma^2 A_0(k)} k^{-2(1+\Gamma)/\Gamma} (7f_2 + 2g_2) - \\ &\quad \frac{4(2-3\Gamma)}{\Gamma^3 A_0^2(k)} k^{2(1+2\Gamma)/\Gamma}, \end{aligned} \quad (46)$$

$$kc_4' = f_2 (kf_2') + \frac{3(2-\Gamma)}{8} (kf_2')^2 + \frac{2(2-3\Gamma)}{\Gamma^2 k^2 A_0(k)} f_2, \quad (47)$$

$$\left[k^{2(\Gamma-1)/\Gamma} (d_4' - f_2 f_2') \right]' = -\frac{16}{\Gamma^2 A_0(k)} k^{-2(1+\Gamma)/\Gamma} f_2, \quad (48)$$

$$ke_4' = \frac{3\Gamma-2}{3\Gamma^2 k^2 A_0(k)} (f_2 - 4g_2) + \frac{4(3\Gamma-2)}{\Gamma^3 k^4 A_0^2(k)}, \quad (49)$$

$$\left[k^{2(\Gamma-1)/\Gamma} f_4' \right]' = \frac{4(2-3\Gamma)}{\Gamma^3 A_0^2(k)} k^{-2(1+2\Gamma)/\Gamma} - \frac{4}{3\Gamma^2 A_0(k)} k^{-2(1+\Gamma)/\Gamma} (f_2 - g_2), \quad (50)$$

and

$$\left[k^{2(\Gamma-1)/\Gamma} g_4' \right]' = \frac{4}{\Gamma^2 A_0(k)} k^{-2(1+\Gamma)/\Gamma} f_2, \quad (51)$$

where

$$\Sigma_4' \equiv b_4' + \frac{\Gamma}{4} k g_2' f_2' + \frac{1}{3} [f_2 (f_2 - g_2)]'. \quad (52)$$

The equations for the time dependent coefficients for ρ and \mathcal{J}_i are obtained from Eqs. (70) and (71) in Appendix A. Also note that we have still not specified the explicit form for $k(t)$.

It is not crucial that we write down here the general solutions to these equations (*i.e.*, the sum of the “homogeneous” and “particular” solutions), even though it is straightforward to integrate and determine their form [24]. The crucial thing is to notice that Eqs. (44), (45), (47), and (49) have only one derivative. In the synchronous gauge approach of Ref. [1] the equations all have two derivatives. Therefore, there are “homogeneous” solutions appearing in the synchronous gauge calculation that do not appear here. For instance, the general solution for $g_2(k)$ is

$$g_2(k) = \gamma_1 - \frac{3}{\Gamma} k^{2(2-3\Gamma)/3\Gamma}, \quad (53)$$

whereas the general solution of Ref. [1] is

$$g_2(t) = \gamma_1 + \gamma_2 t^{-1} - \frac{1}{4} \frac{9\Gamma^2}{9\Gamma - 4} t^{-2(2-3\Gamma)/3\Gamma}. \quad (54)$$

To compare these two we must remove the final arbitrariness left in our solutions by giving an explicit form for $k(t)$. The appropriate choice is $k(t) \propto t^{-1}$ since the zeroth-order solution for the extrinsic curvature obtained by Ref. [1] has a trace given by ${}^{(0)}K \propto t^{-1}$. More generally, since $A_0(k) \propto k^{-4/3\Gamma}$ then the trace of the extrinsic curvature must decay with time if the zeroth-order “expansion factor” A_0 is to grow with time. A trace like $k(t) \propto t^{-1}$ thus results in a power law time dependence for A_0 . It now follows that the time dependence of the “particular” solutions in Eqs. (53) and (54) is the same ($t^{-2(2-3\Gamma)/3\Gamma}$). The synchronous gauge result for g_2 has the extra “homogeneous” solution $\gamma_2 t^{-1}$. However, Comer *et al.* [1] demonstrated a coordinate transformation within the synchronous gauge exists whereby this extra term can be removed. The $K = k(t)$ slicing condition automatically removes the extra term. It is clear that pure gauge terms will automatically be removed at the fourth-order as well.

Thus, as claimed in the Introduction, there are pure gauge terms appearing in the synchronous gauge approach that are not present here. However, we *do* recover the other terms that Ref. [1] obtained. And like the results of Ref. [1], it can be shown [24] that

some of the “homogeneous” solutions (for instance, the γ_1 term in Eq. (53)), can either be absorbed into a redefinition of the “seed” or decay with time for all allowed values of Γ . The other “homogeneous” solutions, as well as the “particular” solutions, decay if $\Gamma < 2/3$, that is, when the matter is ‘inflationary.’ As explained in Ref. [1], this means that the metric perturbations will all freeze out once the comoving scale of the Hubble radius (which is shrinking like $t^{1-2/3\Gamma}$) becomes less than the characteristic scale L associated with the perturbations.

When $\Gamma > 2/3$, then the opposite behaviour happens. Instead of decaying with time, the metric perturbations will all thaw out, and hence grow, as they enter the Hubble radius. In this case, one must keep more and more terms if the solutions are going to be accurate [23]. It is for precisely this reason that the $K = k(t)$ condition, and a non-zero shift vector, can be of great help. The differential equations for the time-dependent coefficients are simpler, and it may be possible to make the expansions converge faster, in the sense that a fewer number of iterations would be required.

5. The Lapse Function Solution

In this section we will verify that the lapse function used in the previous section does indeed satisfy Eq. (8). Recall that we found in Sec. 3 that the spatial gradients of N had to be small. With that in mind, N is expanded like

$$N = N_0(t) + N_2(t)R + N_4(t)R^2 + \eta_4(t)R_i^j R_j^i + n_4(t)D_i D^i R + \dots \quad (55)$$

The equation that determines N is the appropriate form of Eq. (8), setting $N^i = 0$ and using $K = k(t)$:

$$\tilde{D}^i \tilde{D}_i N - 4\pi N (\rho + \mathcal{S}) - N K_i^j K_j^i = -\dot{k}(t) \quad (56)$$

The zeroth-, second-, and fourth-order equations for N are constructed in the same manner as the previous section. (For the reader’s convenience, the expansions to fourth-order for ρ and $K_i^j K_j^i$ are listed in Appendix A.) But unlike the previous section, the equations for $N_0(t)$, $N_2(t)$, *etc.*, are algebraic [24].

The solution for N is

$$N = \frac{2}{\Gamma} \frac{\dot{k}}{k^2} + \frac{2-3\Gamma}{\Gamma^2 k^4 A_0(k)} \dot{k} R - \left[\frac{2-3\Gamma}{\Gamma^2 k^4 A_0(k)} f_2 + \frac{3(2-\Gamma)}{16k^2} (k f_2')^2 \right] \dot{k} R_i^j R_j^i +$$

$$\left[\frac{2-3\Gamma}{3\Gamma^2 k^4 A_0(k)} (f_2 - g_2) + \frac{(2-3\Gamma)^2}{2\Gamma^3 k^6 A_0^2(k)} + \frac{2-\Gamma}{16k^2} (kf'_2)^2 \right] \dot{k} R^2 +$$

$$\left[\frac{2-3\Gamma}{6\Gamma^2 k^4 A_0(k)} (f_2 - 4g_2) + \frac{2(2-3\Gamma)}{\Gamma^3 k^6 A_0^2(k)} \right] \dot{k} D_i D^i R. \quad (57)$$

Note the requirement $N > 0$ implies $\dot{k}/k^2 > 0$. Also notice that Eq. (57) does indeed agree with the result obtained from Eq. (38), making use of Eq. (67) from Appendix A as well as Eqs. (43-51) for the metric corrections.

The time-dependence of N can be qualitatively understood by using again $k(t) \propto t^{-1}$ and then inserting $f_2 \sim g_2 \sim t^{-2(2-3\Gamma)/3\Gamma}$. It is seen that $N_2/N_0 \sim t^{-2(2-3\Gamma)/3\Gamma}$ and $N_4/N_0 \sim \eta_4/N_0 \sim n_4/N_0 \sim t^{-4(2-3\Gamma)/3\Gamma}$. Thus the corrections to N all decay with time when $\Gamma < 2/3$, *i.e.*, the matter is ‘inflationary.’ Hence, there is no contradiction with simultaneously having a monotonic $k(t)$ and a positive lapse function. On the other hand, the corrections all grow if $\Gamma > 2/3$. Hence, one must be cautious in the use of the long wavelength solutions [23] since the corrections to N are not positive-definite. (The second-order term is positive at all times when $\Gamma > 2/3$ if the “seed” is such that $R < 0$.) Certainly, N is positive for the amount of time that the second- and fourth-order corrections remain smaller than the zeroth-order term. After that, the sign of N will depend on the particular value for Γ and the form of the “seed.”

6. Conclusion

The main point of the long wavelength iteration scheme is to provide analytical cosmological models that contain large scale inhomogeneities. These solutions are perturbative and are obtained as expansions in spatial gradients. The fundamental difficulty with this scheme is the lack of an absolute notion of space, and thus an absolute notion of spatial gradient. Any such notion must be introduced by hand. Therefore, it is essential to verify that these long wavelength inhomogeneities result from real physics, and not from embedding unnecessarily wrinkled spacelike slices into an otherwise homogeneous spacetime.

We have addressed this issue by applying the long wavelength iteration scheme to the 3+1 form of the Einstein-Perfect Fluid field equations. We have verified the results of Ref. [1] using the $K = k(t)$ slicing condition, and in the process demonstrated that

the 3+1 formalism makes it easier to implement the long wavelength iteration scheme. In particular, it was seen that a non-zero shift vector can be used to exponentially dampen the effect of anisotropy on the extrinsic curvature and also spurious gauge modes that arise in the synchronous gauge approach do not show up when the $K = k(t)$ slicing condition is invoked.

Even though we have gone a long way in applying the 3+1 formalism to the long wavelength iteration scheme, there are still some questions that need to be considered. For instance, we did not solve the 3+1 equations in the canonical way, *i.e.*, as an initial-value problem. It is clear that this should be investigated, since ultimately the question of how many terms to keep in the expansions should boil down to conditions on the initial data set. The problem [25] can be framed around York’s procedure for handling the constraints [8]. That is, one performs a conformal transformation on the metric, extrinsic curvature, and matter variables, and then uses York’s covariant decomposition of symmetric tensors into their trace-free, vector, and trace parts. There is even a small clue in what has been presented here that this is an appropriate way to proceed: the three-metric naturally had a conformal factor appear, which was the third-root of the determinant of the three-metric. This is precisely the factor suggested by York’s procedure.

Finally, it would also be interesting to see how to construct and use “minimal-strain” or “minimal-distortion” shift vectors [10] in the long wavelength iteration scheme. Smarr and York [10] show that these vectors are very adept at simplifying the form of the three-metric, by separating as completely as possible the purely “kinematical” (in the words of Ref. [10]) from the dynamical parts of the three-metric. This is presently being investigated [25]. If time-independent “minimal-strain” or “minimal-distortion” shift vectors can be constructed, then they can be immediately placed into Eq. (28).

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Appendix A

For the convenience of the reader we list here some basic formulas used to build the second- and fourth-order equations given in the main text. In all that follows consider that the three-metric has the form

$$\gamma_{ij} = {}^{(0)}\gamma_{ij} + \delta\gamma_{ij} , \quad (58)$$

where

$$\delta\gamma_{ij} = {}^{(2)}\gamma_{ij} + {}^{(4)}\gamma_{ij} , \quad (59)$$

and ${}^{(2)}\gamma_{ij}$ and ${}^{(4)}\gamma_{ij}$ are the same as Eqs. (36) and (37), respectively, in the main text.

The first formula is that for the inverse metric γ^{ij} . It is given by

$$\gamma^{ij} = {}^{(0)}\gamma^{ij} + \delta\gamma^{ij} \quad (60)$$

where

$$\delta\gamma^{ij} = -{}^{(0)}\gamma^{ik} {}^{(0)}\gamma^{jl} \delta\gamma_{kl} + {}^{(0)}\gamma^{il} {}^{(0)}\gamma^{jm} {}^{(0)}\gamma^{kn} {}^{(2)}\gamma_{mn} {}^{(2)}\gamma_{kl} . \quad (61)$$

The inverse metric is necessary for constructing the connection coefficients, the Ricci tensor and scalar, and the extrinsic curvature.

Letting

$${}^{(0)}\Gamma_{jk}^i = \frac{1}{2} {}^{(0)}\gamma^{il} \left[{}^{(0)}\gamma_{jl,k} + {}^{(0)}\gamma_{kl,j} - {}^{(0)}\gamma_{jk,l} \right] , \quad (62)$$

then it can be shown [21] that the connection coefficients are

$$\tilde{\Gamma}_{jk}^i = {}^{(0)}\Gamma_{jk}^i + \delta\Gamma_{jk}^i, \quad (63)$$

where

$$\delta\Gamma_{jk}^i = \frac{1}{2} {}^{(0)}\gamma^{il} \left[{}^{(0)}D_k \delta\gamma_{jl} + {}^{(0)}D_j \delta\gamma_{kl} - {}^{(0)}D_l \delta\gamma_{jk} \right] \quad (64)$$

and ${}^{(0)}D_i$ is the covariant derivative compatible with ${}^{(0)}\gamma_{ij}$ (i.e., ${}^{(0)}D_i {}^{(0)}\gamma_{jk} = 0$).

The Ricci tensor is obtained from

$$\tilde{R}_{ij} = {}^{(0)}\tilde{R}_{ij} + {}^{(0)}D_k \delta\Gamma_{ij}^k - {}^{(0)}D_j \delta\Gamma_{ik}^k. \quad (65)$$

In terms of the metric corrections given in Eqs. (44) and (45) then

$$\begin{aligned} \tilde{R}_i^j = & \frac{1}{A_0} R_i^j + \frac{1}{A_0} \left[\frac{1}{2} g_2 \left(D_i D^j R - \frac{1}{3} D_k D^k R \delta_i^j \right) - \frac{1}{2} f_2 D_k D^k \left(R_i^j - \frac{1}{3} R \delta_i^j \right) + \right. \\ & 3f_2 \left(R_k^j R_i^k - \frac{1}{3} R_l^k R_k^l \delta_i^j \right) - \frac{3}{2} f_2 R \left(R_i^j - \frac{1}{3} R \delta_i^j \right) + \frac{1}{6} (f_2 - 4g_2) D_i D^j R - f_2 R_k^j R_i^k + \\ & \left. \frac{1}{3} (f_2 - g_2) R R_i^j \right]. \end{aligned} \quad (66)$$

Derivatives of the determinant of the three-metric are obtained from $\delta\gamma = \gamma\gamma^{ij}\delta\gamma_{ij}$.

Letting δ be $\partial/\partial t$, then

$$\begin{aligned} \frac{\dot{\sqrt{\gamma}}}{\sqrt{\gamma}} = & 3H_0 + \frac{1}{2} \dot{g}_2 R + \frac{1}{2} \left(\dot{c}_4 - f_2 \dot{f}_2 \right) R_l^k R_k^l + \frac{1}{2} \left(\dot{a}_4 + \frac{1}{3} [f_2 \dot{f}_2 - g_2 \dot{g}_2] \right) R^2 + \\ & \frac{1}{2} \dot{c}_4 D_k D^k R, \end{aligned} \quad (67)$$

where $H_0 = \dot{A}_0(t)/2A_0(t)$. Notice that it is only $g_2(t)$ which enters as a second-order term.

The extrinsic curvature is

$$\begin{aligned} K_i^j = & \frac{1}{3} k(t) \delta_i^j + \frac{k(t)}{6H_0} \left[\dot{f}_2 \left(R_i^j - \frac{1}{3} R \delta_i^j \right) + \left(\dot{b}_4 - \frac{1}{6H_0} \dot{f}_2 \dot{g}_2 + \frac{1}{3} [f_2 (f_2 - g_2)] \right) \times \right. \\ & R \left(R_i^j - \frac{1}{3} R \delta_i^j \right) + \left(\dot{d}_4 - f_2 \dot{f}_2 \right) \left(R_k^j R_i^k - \frac{1}{3} R_l^k R_k^l \delta_i^j \right) + \dot{f}_4 \left(D_i D^j R - \frac{1}{3} D_k D^k R \right) + \\ & \left. \dot{g}_4 D_k D^k \left(R_i^j - \frac{1}{3} R \delta_i^j \right) \right]. \end{aligned} \quad (68)$$

Notice that the correcting pieces are such that we still maintain $K = K_i^i = k(t)$. It is also found that

$$K_l^k K_k^l = \frac{1}{3}k^2(t) + \frac{k^2(t)}{(6H_0)^2} \dot{f}_2^2 \left(R_l^k R_k^l - \frac{1}{3}R^2 \right). \quad (69)$$

This has corrections beginning with the fourth-order. In the synchronous gauge, there are corrections starting with the second-order. However, it is really because $K = k(t)$ holds at *all* orders that our second- and fourth-order equations are simpler than those of the synchronous gauge.

Finally we list the equations—good to fourth-order—for ρ and \mathcal{J}_i :

$$\begin{aligned} 16\pi\rho = & \frac{2}{3}k^2(t) + \frac{1}{A_0}R + \frac{1}{6A_0} [f_2 - 4g_2] D_i D^i R - \left[\frac{1}{A_0}f_2 + \frac{k^2(t)}{(6H_0)^2} \dot{f}_2^2 \right] R_i^j R_j^i + \\ & \frac{1}{3} \left[\frac{1}{A_0} (f_2 - g_2) + \frac{k^2(t)}{(6H_0)^2} \dot{f}_2^2 \right] R^2 \end{aligned} \quad (70)$$

and

$$\begin{aligned} \frac{48\pi H_0}{k(t)} \mathcal{J}_i = & \frac{1}{6} \dot{f}_2 D_i R + \frac{1}{6} \left[3\dot{g}_4 - \dot{b}_4 + \frac{1}{6H_0} \dot{f}_2 \dot{g}_2 - \frac{1}{3} (f_2 [f_2 - g_2]) \right] R D_i R + \\ & \left[\dot{b}_4 + \frac{1}{3} \dot{f}_4 - \frac{1}{2} \dot{g}_4 + \frac{1}{2} (\dot{d}_4 - f_2 \dot{f}_2) - \frac{1}{6H_0} \dot{f}_2 \dot{g}_2 + \frac{1}{3} (f_2 [f_2 - g_2]) \right] R_i^j D_j R + \\ & \left[\dot{d}_4 - f_2 \dot{f}_2 + 4\dot{g}_4 \right] R_k^j D_j R_i^k - \left[3\dot{g}_4 + \frac{2}{3} (\dot{d}_4 - f_2 \dot{f}_2) \right] R_k^j D_i R_j^k + \\ & \frac{1}{3} \left[2\dot{f}_4 + \frac{1}{2} \dot{g}_4 \right] D_k D^k D_i R. \end{aligned} \quad (71)$$

From this we see $\mathcal{J}_i \mathcal{J}_j$ has six spatial gradients. Hence, $\tilde{u}_i \tilde{u}_j$ can be ignored in all equations. In particular, $\rho \approx \rho^*$ and $\mathcal{S}_i^j \approx (\Gamma - 1) \rho \delta_i^j$.

Appendix B

In this appendix we show how to solve Eq. (23) for ${}^{(0)}\mathcal{K}_i^j$. Let

$$\mathcal{M}^{jk}{}_{il} \equiv \delta_l^j \tilde{D}_i N^k - \delta_i^k \tilde{D}_l N^j \quad (72)$$

and $E^{ai}{}_j$ represent ‘ a ’ eigenvectors with eigenvalues $\lambda_{(a)}$, $a = 1, 2, 3, \dots, 9$; that is,

$$\mathcal{M}^{jk}{}_{il} E^{ai}{}_j = \lambda_{(a)} E^{ai}{}_j, \quad (73)$$

with no sum being done over a . Also let $(E^{-1})_{ai}{}^j$ be such that

$$E^{ai}{}_j (E^{-1})_{bi}{}^j = \delta_b^a \quad \text{and} \quad \sum_a E^{aj}{}_i (E^{-1})_{al}{}^k = \delta_i^k \delta_l^j . \quad (74)$$

Then, the matrix with components $\mathcal{M}^{jk}{}_{il}$ can be diagonalized via

$$E^{ai}{}_j \mathcal{M}^{jk}{}_{il} (E^{-1})_{bl}{}^k = \lambda_{(a)} \delta_b^a . \quad (75)$$

Finally, let

$$\mathcal{K}^a \equiv E^{ai}{}_j {}^{(0)}\mathcal{K}_i^j \quad \text{and} \quad {}^{(0)}\mathcal{K}_i^j \equiv \sum_a (E^{-1})_{ai}{}^j \mathcal{K}^a . \quad (76)$$

Therefore, Eq. (23) becomes

$$\dot{\mathcal{K}}^a - \left(\tilde{D}_k N^k \right) \mathcal{K}^a - \lambda_{(a)} \mathcal{K}^a = 0 \quad (77)$$

and the solutions are thus

$$\mathcal{K}^a = \exp \left[(D_k N^k + \lambda_{(a)}) t \right] \mathcal{C}^a , \quad (78)$$

with \mathcal{C}^a only depending on x^i . Or, letting

$$\mathcal{G}_i^j \equiv \sum_a \exp(\lambda_{(a)} t) (E^{-1})_{ai}{}^j \mathcal{C}^a \quad , \quad \mathcal{G}_j^j = 0 , \quad (79)$$

we get the solution written in Eq. (28).